THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and forty-second regular meeting of the Society was held in New York City on Saturday, February 27, 1909. A single session sufficed for the brief programme. The attendance included the following twenty-four members:

Professor Maxime Bôcher, Professor E. W. Brown, Professor F. N. Cole, Miss E. B. Cowley, Miss L. E. Cummings, Professor H. B. Fine, Dr. Frank Irwin, Professor Edward Kasner, Mr. W. C. Krathwohl, Mr. H. F. MacNeish, Professor W. F. Osgood, Professor R. W. Prentiss, Mr. H. W. Reddick, Professor L. W. Reid, Mr. L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Dr. Elijah Swift, Professor H. D. Thompson, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. Ç. B. Walsh, Professor H. S. White, Miss E. C. Williams.

The President of the Society, Professor Maxime Bôcher, occupied the chair. The Council announced the election of the following persons to membership in the Society: Mr. W. T. Campbell, Boston Latin School; Professor W. A. Garrison, Union College; Mr. D. D. Leib, Johns Hopkins University; Professor William Marshall, Purdue University; Mr. J. B. Smith, Richmond, Va., High School; Mr. C. M. Sparrow, Johns Hopkins University. Ten applications for membership in the Society were received.

Professor Bôcher presented his resignation as member of the Editorial Committee of the *Transactions*, to take effect August 15, it being his intention to spend the coming year abroad. Professor Osgood was appointed to fill Professor Bôcher's unexpired term.

The following papers were read at this meeting:

(1) Professor EDWARD KASNER: "Brachistochrones and tautochrones."

(2) Dr. D. C. GILLESPIE: "On extremal curves which are invariant under a continuous point transformation group."

(3) Professor VIRGIL SNYDER: "Infinite discontinuous groups of birational transformations which leave certain surfaces invariant."

Professor Bôcher presented certain results in extension of his paper "On systems of linear differential equations of the first order," read February 22, 1902, and to be published in an early number of the *Transactions*.

In the absence of the authors the papers of Dr. Gillespie and Professor Snyder were read by title. Professor Kasner's paper will appear in the BULLETIN. Abstracts of the other papers follow below, being numbered to correspond to the titles in the list above.

2. Dr. Gillespie studies the connection between an integral, its extremal curves, and a continuous group of point transformations under which they may be invariant.

The condition that a two-parameter family of curves in the plane be the extremal curves of an integral is expressed by a partial differential equation of the first order (a), which the integrand must satisfy.* This system of extremals is the solution of an ordinary differential equation of the second order (b). If the extremals are invariant under a transformation group, there exists at least one first integral of the equation (b) which is invariant under the transformation; *i. e.*, every one parameeter family of curves obtained by assigning a value to the arbitrary constant in this first integral is invariant under the transformation.† This first integral together with the transformation defines a solution of the equation (a) which contains an arbitrary function, but which is not necessarily the complete solution of the equation. The relation between this first integral, the transformation, and the system of solutions of (a) can be expressed by an equation. When the first integral is one of the knowns, this is an algebraic equation; when the first integral is the unknown, it is a differential equation, which requires nevertheless only a quadrature for its solution.

3. Only a few isolated examples of surfaces have been found which are invariant under a discontinuous group of birational transformations of infinite order. Professor Snyder shows that these examples all depend upon a certain (2, 2) correspondence. Three interpretations are given. The first applies to quartic surfaces having more than one conical point. In any plane

* Bolza, Lectures on the calculus of variations, p. 31.

† Lie-Scheffers, Vorlesungen über Differentialgleichungen, p. 375. Page, Ordinary differential equations, p. 149.

through two conical points the (2, 2) correspondence is defined by the pencils through the nodes. The second also refers to quartic surfaces, those having two nets of hyperelliptic curves. The (2, 2) correspondence is defined by the lines joining the points of the canonical g_2^1. The third concerns the systems of bitangents on any surface which is complete focal surface of two or more congruences.

F. N. COLE,
Secretary.

BÉZOUT'S THEORY OF RESULTANTS AND ITS INFLUENCE ON GEOMETRY.

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BY PRESIDENT HENRY S. WHITE.

THE accepted truths of today, even the commonplace truths of any science, were the doubtful or the novel theories of yesterday. Some indeed of prime importance were long esteemed of slight importance and almost forgotten. The first effect of reading in the history of science is a naive astonishment at the darkness of past centuries, but the ultimate effect is a fervent admiration for the progress achieved by former generations, for the triumphs of persistence and of genius. The easy credulity with which a young student supposes that of course every algebraic equation must have a root gives place finally to a delight in the slow conquest of the realm of imaginary numbers, and in the youthful genius of a Gauss who could demonstrate this once obscure fundamental proposition.

The first complete proof, by Gauss, that rational algebraic equations have roots either real or imaginary dates back only to 1799. That part of algebra that is concerned with equations is accordingly for the most part modern, recent indeed, as compared for instance with the plane geometry of lines and circles. Before Gauss, it is true, much had been done in actually solving equations of the lower orders and in the theory of symmetric functions of the roots. After him also the question of the arithmetical character of the roots required not only a Galois to penetrate its mystery, but also a Liouville and a Jordan to expound the marvellous theory that Galois had created.

In the general mathematical consciousness of the present day is dominant, even when not active, the knowledge revealed by Gauss and Galois; and this widespread understanding of the nature of the solutions of a single equation constitutes the goal toward which tended for three centuries the labors of Tartaglia, Cardan, Ferrari, and the long train of emulous scholars culminating in Lagrange and Abel. The theory of the single equation in one unknown is now, if not completed in all its details, yet at least finished in foundation and in the framework of its superstructure.

Equations in two variables have received no less attention, but from the nature of the case it must be long before their theory could reach any corresponding stage of completeness. Geometrically, a set of discrete points on a line has obviously fewer properties to investigate than a curve in a plane. The projective covariants of the one are all, like itself, sets of discrete points; while those of the other are of four or more different kinds: points, lines, curves, and line-loci or envelopes. Of a binary form, or an ordinary equation in one unknown, one seems to have a satisfactory picture in a set of a definite number of points on a line; but in endeavoring to apprehend even the picture of a plane curve, one begins instinctively to look for inflexional tangents, bitangents, and soon for polars, Hessian, Cayleyan, and other auxiliary loci; so that an algebraic curve becomes to our understanding an associated host of curves, even before we rise to the concept of a class of Riemann surfaces. And in fact, by reflex influence, since the developments of the past forty or fifty years, the geometry of sets of points on a line is now concerned less with the individual set, but much more with the related infinite linear systems or involutions.

Hence simultaneous equations were necessarily to be discussed, if geometry was to develop in the algebraic direction foreshadowed by Descartes and Newton. Descartes it was who proposed to classify plane curves as algebraic and not algebraic, and to divide the former according to the degree of their equations. Analysis quickly overtook pure geometry in developing the properties of loci of the second order, and it was easy for Euler to prove that the equation for the intersection of two second order loci is of degree 4. The next step in generalization was to determine the degree of the eliminant, or equation for the common points satisfying two simultaneous equations of any orders, m and n. This question was solved independently

in 1764 by Euler, whose interest in mathematics was universal, and by a young French student who seems to have confined himself exclusively to this narrow field of research, Etienne Bézout. Both gave the degree as $m \cdot n$, the product of the orders of the intersecting loci, and both proved the theorem by reducing the problem to one of elimination from an auxiliary set of linear equations. Both, that is, depended upon the formal structure of what were later named *determinants*. The result of this initial publication, restricted to two equations, is what has kept current the fame of Bézout, and the determinant resulting from his method is what Sylvester and later writers call the Bézoutiant. But this was to Bézout only the beginning of his lifelong study in the formation of eliminants.

The mode of formation of this resultant of two equations in one unknown, or eliminant of two in two variables, is familiar to most students; Brill and Noether show that an equivalent process was employed by Newton for forming resultants of equations of low degrees, though without the almost obvious extension to equations in two unknowns. Two things about it were patent as clues to generalization. First, it gave a direct process for combining any two equations to eliminate any one unknown. Second, the resultant when found is a linear combination of the two original functions, with multipliers that are rational in the variables and in the coefficients of the given functions,

$$R \equiv F_1 f_1 + F_2 f_2.$$

With more equations and more variables, which of these two features would be more useful? We must remember that in 1765 not even the degree of the eliminant was known, and that the chief effect desired in a scheme for removing two or more unknowns from a set of equations was that it should indicate the degree of the result (e. g., the exact number of intersections of three surfaces of given orders).

The use of a direct process, to be applied step by step in some systematic sequence, is certainly the more tempting; and this mode has been employed very extensively of late — the highest common factor process — as affording more cogent deductive arguments at every stage. But it has its difficulties, as Bézout indeed found out; for he experimented many years before he found the more feasible plan. If one determines to use it, as does Kronecker for example, one must discriminate

with care between an essential factor and an adventitious factor in every resultant, and learn to calculate in advance the degree of each. Bézout chose finally the other horn of the dilemma, and availed himself of the right to use hypothesis; he ventured to guess that the other style of attack would be successful.

It was his hypothesis that from $k+1$ equations k variables could be eliminated at one stroke; and that the result would be a linear combination of the given functions. This latter is equivalent to the famous Fundamental-Satz of Noether, and its proof is usually based upon the existence and the particular degree of the resultant. So we recognize the venturesome character of this attack. Supposing, then, that by using multipliers F_i of degree sufficiently high he could produce a combination

$$R_n \equiv F_{m_1} f_{n1} + F_{m_2} f_{n2} + \cdots + F_{m_k} f_{nk} + F_{m_{k+1}} f_{n_{k+1}}$$

that should be free from k variables or unknowns, say

$$x_1,\ x_2,\ \cdots,\ x_k$$

and contain only x_{k+1}, he set out to determine the necessary value of n, the degree of R in the sole remaining unknown x_{k+1}. That is to say, he sought for a minimum value n which should be sufficiently great for the purpose.

Naturally his work has received many improvements from later workers; and we must recognize the large credit due to Professor Eugen Netto for having worked out so many supplementary lemmas in his admirable Vorlesungen über Algebra. To rescue a theory from its own weaknesses is no less deserving of honor than the invention of new theories. Only by a vast amount of such patient self-forgetful devotion to the interests of science can the body of demonstrated truth grow as a solid and unshakable structure. It will not injuriously misrepresent Bézout therefore if we say that he considered all the coefficients in the multipliers $F_{m_1}, F_{m_2}, \cdots, F_{m_{k+1}}$ as undetermined arbitrary parameters, and inquired first *how many of them must remain arbitrary after the resultant R_n has been completely determined.* Obviously any two terms in the combination could be modified together, since identically

$$F_\alpha f_a + F_\beta f_b \equiv (F_\alpha + \phi_{a,b} \cdot f_b) f_a + (F_\beta - \phi_{a,b} \cdot f_a) f_b,$$

so that the coefficients in all such functions $\phi_{a,b}$ will remain *essentially indeterminate.* But again, in this way certain modify-

ing terms are enumerated more than once (as those containing the product $f_a \cdot f_b \cdot f_c$, etc.). Hence was necessary a considerable invention and application of a calculus of finite differences to solve the enumerative problem: this forms the first part of Bézout's great work; and this is reproduced well by Serret (Algèbre supérieure) and Netto.

Excluding essential indeterminates, call the number of *available* undetermined parameters P, or say $P(n, k+1)$; and let $N(n, k+1)$ denote the number of terms that can occur in a polynomial of order n in $k+1$ variables. Since $n+1$ terms in x_{k+1} are expected in R, the number to be removed is $N(n, k+1) - n$. Hence the inequality to be satisfied, stating that there are no more terms to be excluded than there are available arbitrary multipliers, is

$$P(n, n_1, n_2, \cdots, k+1) - 1 \geqq N(n, k+1) - n - 1$$

or

$$N - P \leqq n.$$

If R is to be a determinate function, this shows that its degree must be exactly $N - P$. But as n increases from the largest of the numbers $n_1, n_2, \cdots, n_{k+1}$, $N - P$ reaches soon its maximum value and then, as a constant, remains superior in magnitude until n becomes equal to the product $n_1 \cdot n_2 \cdot n_3 \cdots n_{k+1}$. From that point on of course the inequality is reversed; that is, the parameters are more numerous than the conditions to be satisfied. Hence *the degree of any completely determinate resultant must be exactly the product of the degrees of the original equations.*

This leaves unsettled of course first the question of the *existence* of such a resultant, namely, a determinate linear combination of the given functions with rational multipliers; in other words the question whether for $n = N - P$ the linear equations to be satisfied are all independent and consistent. Also, secondly, it leaves to be fixed by further consideration the choice of terms whose coefficients shall be equated to zero, in order from the linear equations thus formed to eliminate the auxiliary coefficients by means already familiar and so to obtain the eliminant expressed as a determinant. On this second point no one has yet succeeded in elaborating such a scheme for substitution as evidently Bézout supposed was practicable. We shall find his own words on this point not critically precise, but interesting as showing how loosely constructed arguments once passed

current. "But if we imagine that the first equation is multiplied by a complete polynomial of order m in the same number of unknowns, and that in the resulting equation of degree $m_1 + n_1$ (the 'product equation') there are substituted in all the terms where it is possible to do this the value of $x_1^{n_1}$, that of $x_2^{n_2}$, that of $x_3^{n_3}$, etc., then as the multiplier polynomials will have brought into the product equation as many different coefficients as there are terms, we see that after those substitutions there can remain of terms in x_1, x_2, x_3, etc., only so many as it will be possible to make disappear by the help of the coefficients in the polynomial multipliers." Further on, more to the same effect about these substitutions, but nowhere an examination into the difficulties of such substitutions when more than one quantity is to have its exponent reduced.

No such difficulty arises if only a very special set of equations is considered, namely, that in which each equation in its turn contains only the first power of its correspondingly numbered unknown quantity together with other terms involving later unknowns but none earlier,

$$\begin{aligned} f_{n_1} &= x_1 - g_{n_1}(x_2, x_3, x_4, \cdots, x_{k+1}) = 0, \\ f_{n_2} &= x_2 - g_{n_2}(x_3, x_4, \cdots, x_{k+1}) = 0, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ f_{n_k} &= x_k - g_k(x_{k+1}) = 0, \end{aligned}$$

while the last may be of any form $f_{n_{k+1}}(x_1, x_2, \cdots, x_k, x_{k+1}) = 0$. What Bézout very properly terms substitution is described more fully by precisians of the present century as the addition of the function f_i multiplied by a suitable polynomial; since a term like ax_1^r for instance is replaced by adding to it the product

$$a(x_1 - g_{n_1}) \cdot \left\{ \frac{x_1^r - g_{n_1}^r}{x_1 - g_{n_1}} \right\} \equiv a(x_1^r - g_{n_1}^r).$$

In this example, an unknown once removed from $f_{n_{k+1}}(x_1, x_2, \cdots, x_k, x_{k+1})$ remains absent, and the first k of them being removed thus from the $(k+1)$th, there remains the eliminant in the single unknown x_{k+1} to the proper degree.

But for the general "complete" equations, the attempt to construct the elimination process *progressively* after this model has not met with success hitherto, since an unknown after having its degree lowered by an early substitution may have it raised by a later substitution. Even Netto here attempts

nothing further than a demonstration by the aid of linear equations. The general case of course would be not the very simple example cited above; each variable would appear in each to several different powers, and direct substitution would lower the degree of one variable only; the aim being to depress the degree to a maximum $n_r - 1$ for x_r $(r < k + 1)$. If this is possible, it gives what may be called for certain purposes a normal form.

Now this is a topic of scarcely less interest and importance than the matter of elimination itself — the reduction of a polynomial, by the aid of a set of equations, to another form in which the variables shall appear with exponents not higher than prescribed limiting values. This is for Bézout a transition form, from which finally the coefficients of all terms containing the variables to be removed are set equal to zero, giving the linear system needed for the closing step of his process of elimination. But as the form is unique, it should prove and has proved equally interesting, as a reduced normal form in a modular system, with the other reduced normal form that we call *the* eliminant.

There were still questions to be investigated when Bézout laid down his pen and published his *magnum opus* in 1779; but certainly he did not leave the subject where he found it. Where he left it, Cayley was to resume it later and at one stroke express a resultant not indeed by a single determinant, but by a fraction whose numerator and denominator contain only determinant factors, found rationally from the coefficients of the given equations. Bézout's system of equations is made to yield a definitive result when once the distinction of available and non-available parameters is abrogated, and all are treated alike. And as for the *degree* of the eliminant, the main problem in his time, Bézout had fixed that both for the so-called general case, and also for a large number of particular cases. One may say that he determined the number of finite intersections of algebraic loci, not only when all the intersections are finite, but also when singular points, or singular lines, planes, etc., at infinity occasion the withdrawal to infinity of certain of the intersection points; and this at a time when the nature of such singularities had not been developed.

All algebraic geometry is a reduction, actual or possible, of systems of equations to a different form, plus the specification of what the symbols shall denote. The resultant being once known, of course its interpretation will thereafter be found

often in geometric discussions. To survey the applications of the eliminant therefore is practically to enumerate the different chapters that have been created in algebraic geometry.

The first would certainly be what is called Cramer's paradox. This antedates Bézout, for Cramer's book appeared in 1750, and may probably have helped to form Bézout's opinion on the urgent need for a solution of his problem. The paradox is, briefly, this. The equation of a plane curve of order n contains $\frac{1}{2}(n+1)(n+2)$ terms, depends therefore on the same number of coefficients, less one. To require the curve to pass through that number of given points is then, algebraically speaking, to determine the coefficients by means of linear equations, hence *uniquely*. But two curves of that order have in common more than that number of points, namely, n^2 (according to the Bézout theorem). But $n^2 > \frac{1}{2}(n^2+3n)$ if $n > 3$, so that two curves, indeed an infinite pencil of curves of order n can be passed through even more points than would commonly suffice to determine a single curve. This not intricate puzzle was resolved by Euler, leading to the interesting and important notion of interdependent sets of points in a plane. From this concept was to arise later that of involutions of points in a plane (or in space of more than two dimensions), and thus was opened a field which has only been entered, but not yet explored. An involution is generated when in a set of points of intersection of two (or three) curves enough points are fixed to leave only one still arbitrarily variable, while $r-1$ others vary with it. This one then being supposed to describe the whole plane, the totality of positions of the whole set of r varying points, a doubly infinite system of point sets, constitutes an *involution* in the plane.

Next in order of development is the notion of multiple points of curves. Bézout treated them by implication only, locating them at infinity, and showed for many cases how they affect the number of finite intersections. These questions were taken up, five decades later, by Plücker, and led to the discovery of the universally known Plücker relations that connect the numbers of double points, inflexional points, double tangents, and cusps, of a plane curve. These latter gave to geometry the notion of the *deficiency* of a curve, that notion which was to be analyzed to its depths by Riemann and by Clebsch, and to form the chief organic bond between algebraic curves and the theory of multiply periodic functions of a complex variable.

But Cramer's paradox led naturally also to the examination of the sets of variable intersections cut out upon a curve of one order by a varying curve of different order; that is, to the questions of curves having contacts one or many on the fixed curve; of the degree of freedom in a variable point set on the curve; in short to all those questions which lead through Noether's Fundamental-theorem. Before Noether there were theorems in plenty upon the partition of the intersections of two curves into two sets, with the hypothesis that one partial set lie all on one curve of lower order, and the conclusion that the remaining partial set must lie on another curve of complementary order. The best known of these are ascribed to Plücker, Jacobi, and Cayley. But Noether was the first to formulate precisely and prove what was tacitly assumed in them all, that a curve containing all the intersections of two others, $f_1 = 0$ and $f_2 = 0$, can always be represented in the form $F_1 f_1 + F_2 f_2 = 0$, where F_1 and F_2 are rational in the coordinates. One sees that this form is that in which Bézout assumed the resultant, and which he in effect established. Very naturally therefore Noether's proof begins upon that basis, and lays down conditions which while not demonstrably *necessary*, are certainly *sufficient* for the conclusion. Upon this basis then rises the extensive body of theorems relating to adjoint and non-adjoint curves, and in particular the elegant theorem on residual and corresidual point sets on a curve: *If a point set A is corresidual with another B in respect to any third set C, then any fourth set residual to A is residual also to B;* two residual sets constituting, together with the singular points of the basal curve, its complete intersection with an adjoint curve.

Bézout's discussions had regard altogether to the elimination from numerical equations; for in his day the theory of forms, of invariants under the projective group, was not yet known. When that theory arose, the interest of algebra in the resultant was no less keen than that of geometry in the eliminant. Hence in the plane the result of eliminating both coordinates from three equations was examined, an invariant whose vanishing gives the condition for three curves to have a point in common. But three curves might have two, or three, or more, points in common. Cayley therefore proposed the question: If all the first polars of a curve have d points in common, what will be the degree, in the coefficients, of the condition for the occurrence of one further intersection? His answer was $3(n-1)^2 - 7d$, n

being the order of the given curve. More general was Brill's inquiry: When three curves of orders n_1, n_2, n_3 have d points of intersection, what is the condition for the occurrence of one further common point? This question he answers; and so was instituted the theory, yet in its infancy, of *reduced discriminants* and *reduced resultants*.

Pausing briefly in this review of the potentialities of the theorem on the degree of the resultant, though we have spoken so far only of resultants of two forms, let us observe Bézout's own view of the importance of his investigations. In the dedication he says of the work: "It has as its aim the completion of one part of the science of mathematics, on which all the other parts are waiting for that which can at this time secure their own advancement." Then in his preface, after deriding those who had been discouraged from researches barely begun by the complexity of the algebraic relations which they encountered, he continues as follows. "The analysis of infinitesimals, equally attractive and important, . . . has drawn away all the interest and all the toil, and the algebraic analysis of finite quantities, to start from that epoch, seems to have been looked at only as a field in which either there remained nothing further to be done, or else whatever was left to do would have proved fruitless speculation. . . . If we note carefully the fact that in reference to the countless number of equations and of unknowns, upon which the solution of any problem may depend, we know as yet how to treat only the case of two equations and two unknowns; that we understand, I repeat, how to treat only that single case with the certainty of introducing nothing extraneous to the question, then we shall doubtless agree that in this matter everything is yet to be done."

After explaining his methods in 463 pages, Bézout sums up the outcome thus (page xxi). "We think it possible to state that there is no kind of algebraic equations for which we have not given the means of determining the lowest possible degree of the final equation; either when there are or when there are not relations between the coefficients which could occasion a particular lowering of that degree." This belief may be a trifle optimistic, at least it is to be hoped that there will be found means of reaching *enumerative* results with less labor than Bézout's methods would impose. But surely the aim was high, and worthy the devotion of one life time; and the conclusion of his preface must certainly excite our admiration and respect.

"We hope that this work may prove the occasion of great progress in analysis, by turning toward that important field the talent and the cleverness of analysts of our time. We shall regard ourselves as fortunate if on considering the point where we take up these questions and the point where we leave them, it shall be found that we have discharged some part of that tribute which every man owes to society in that state in which he happens to be placed."

A life of unremitting labor is not ill spent if it leaves a work so easily intelligible, so full of interesting problems, and in proportion to contemporary science so complete as this Théorie générale des équations algébriques of Bézout. Yet what a commentary on the futility of the best efforts is found in the fact that both Jacobi and Minding, only 60 years later, published investigations as new whose methods and results were in effect identical with Bézout's! At least this showed not that his work was unnecessary, but only that he was in advance of his time. Was it perhaps that geometry was waiting for Gauss to prove that the degree of an equation indicates actual and not illusory roots? Or was it waiting for Monge and Poncelet and Plücker to set it free from the restraints imposed by a particular system of coordinates? Or perhaps for Liouville or Poisson to adjoin the linear function of all the coordinates with as many indeterminates: $z = k_1x_1 + k_2x_2 + \cdots + k_{n+1}x_{n+1}$, and to discuss not the resultant or eliminant in a single x_i, but the eliminant in z containing all the indeterminates $k_1, k_2, \cdots, k_{n+1}$? The algebraic projective geometry of the present has grown and flourished upon all these preparations; and it appears that not the single isolated worker but the cooperation of a large number, with the utmost facility of communication of results, is needful for the rapid advancement of any branch of mathematics.

To return to the geometrical problems that grow out of the theory of eliminants. In three dimensions, three surfaces intersect in points, real or imaginary, whose number is the product of their three orders. Three quadric surfaces meet in 8 points. Here, as in the plane, arises the question of the interdependence of these points: Cramer's paradox is applicable to space of any number of dimensions, and indeed to intersection systems which are not all points. Of these 8 intersections of three quadrics, if seven are given the eighth can be constructed linearly; and it is surprising to see how many eminent geome-

ters have found in this one problem points of difficulty worthy of their attention. But an eliminant in more than two dimensions, *i. e.*, an intersection system of three or more loci, has a further peculiarly interesting possibility.

Three surfaces may have not only a finite set of points in common, but equally well also a whole curve or system of curves. That is to say, their eliminant may vanish identically, having an infinite number of roots. Three quadric surfaces can have in common a line, two lines, a conic, three lines, or a twisted cubic curve. What would then be the number of additional discrete points of intersection? If there is a common conic C, for example, two of the surfaces meet in C and a second conic K, which have two points in common; and the third surface must cut K in 4 points, the two upon C, and therefore only *two* outside the common conic C. Or speaking briefly, a common conic lying on three quadric surfaces *absorbs* six of their eight intersections. Similarly, a common line absorbs 4, a twisted cubic curve all 8, of the points of intersection of three quadric surfaces. This line of problems is not without difficulty, and seems not to have come to the notice of Bézout except where the common curves were straight lines at infinity; and even here he does not put the description into geometric language.

Evidently there was need of a theory of twisted curves in three dimensions; of gauche surfaces and curves in four dimensions, etc. Cases were known in which not 3, but 4, surfaces were required to define an intersection curve. For this case however there was in Bézout's eliminant a suggestion of a useful mode of attack. By eliminating one variable from two equations one has as eliminant the equation of a cone with vertex at infinity. Treat this in homogeneous coordinates, and we have a cone with vertex at any point, and containing the curve in such a way that on each generator there lies a single point of the curve. A different way of conducting the elimination suggested to Cayley the surface called, since his invention, a monoid; and both Noether and Halphen found the cone-and-monoid a sufficient representative of any twisted curve in 3 dimensions. But of curves and surfaces in projective space of four or more dimensions, I believe that no one has yet worked out a systematic list.

It is to be expected that fundamental propositions will gradually become obscured by their derivatives. When once a

theorem like Noether's has been discovered, which embodies the geometrically most available substance of the Bézout theorem, and when this has found expression in geometric propositions on point sets, residues, and groups of point sets on a curve, then it will be but infrequently that geometers will have recourse to the formation of resultants or eliminants. Yet it is not well that such a basal theorem together with the scheme of operation that it involves should again fall into oblivion. For even at this late date one may read in a quite recently issued American cyclopedia the statement that, in complicated equations, elimination becomes difficult and often impossible. And even in Cayley's article in the Encyclopædia Britannica one reads the more temperate verdict that there does not yet exist a distinct theory of systems of equations! Such statements in what purport to be standard popular works of reference are more often read and credited than the precise summaries given in technical works or even the new mathematical Encyklopädie.

Two relatively recent essays employing resultants and discriminants for the extension of the theory of curves are by Franz Meyer, in the *Mathematische Annalen*, volumes 38 and 43. They proceed from the postulate that truths purely numerical can be established for the kind of curves called rational, and will then hold true for non-rational curves of the same order. The first deals with the ordinary singularities of plane curves; the second, with those of twisted curves in space, determining what consequences ensue to other singularities when two of any one sort come to coincide: *e. g.*, how many inflexions come together as two contacts of a double tangent merge into a hyperosculation point. The immediate aim is to find relations among the real singularities as distinguished from the imaginary ones; but the postulate seems valid and susceptible of adaptation to other uses.

It is not proper to close this sketch without alluding to Kronecker's work in formulating a systematic theory of systems of equations. Under the head of *Modular Systems* he instituted inquiries far beyond the single question of eliminants which filled Bézout's horizon. Some time ago allusion was made to one reduced form of equal significance with the eliminant, where all the variables were retained, but with exponents so lowered that the reduced form was uniquely determined. The study of all such reduced forms under any given system of equations (or moduli), whether general or special, is part of Kronecker's pro-

gramme, and in particular the grouping together of all forms, which, like the resultant, are reducible to zero by the aid of given equations, under the class name of an algebraic modulus. In his Festschrift and the later expository papers of his pupils are proposed methods for testing any given system for its character, whether general, or special of the first sort (loci with a curve in common), or of the second or higher sort (loci with a surface, etc., in common). The expansion of this body of doctrine or abstract theory into a concrete *geometry* with fulness of examples remains a task, not all deductive but largely creative, for coming decades or generations.

Not the possession of eliminants actually calculated by Bézout's deservedly famous scheme is needful for the geometer, but the knowledge of the conditions under which such an eliminant will exist, and what conditions will modify it. So with regard to the more far-reaching scheme of Kronecker; it is ultimately, perhaps, not the full elaboration of particular examples as such, that we wish to have, but a precise knowledge of *how* the relative operations could be executed in finite time, and a precise formulation of conditions that would modify or influence the result of those operations. Which is of greater value, the logic or the concrete object to which it is applied? Let everyone decide when both are in his possession!

ON THE REPRESENTATION OF NUMBERS BY MODULAR FORMS.

BY PROFESSOR L. E. DICKSON.

(Read before the Chicago Section of the American Mathematical Society, January 2, 1909.)

1. For any field F in which there is an irreducible equation $f(\rho) = 0$ of degree m, the norm of

$$x_0 + x_1\rho + x_2\rho^2 + \cdots + x_{m-1}\rho^{m-1}$$

is a form of degree m in m variables which vanishes for no set of values x_i in the field F, other than the set in which every $x_i = 0$. For a finite field it seems to be true that every form of degree m in $m+1$ variables vanishes for values, not all zero, in the field. For $m = 2$ and $m = 3$ this theorem is

established in §§ 2, 3. The corresponding theorem does not hold in general for infinite fields. But A. Meyer * has shown that any indefinite quadratic form in five variables vanishes for integral values, not all zero, of the variables.

Modular forms which represent only squares have been investigated † at length by the writer; those which represent cubes exclusively are considered in §§ 4–12.

2. Any ternary quadratic form in the $GF[p^n]$, $p > 2$, can be transformed into $q = ax^2 + by^2 + cz^2$. The number of sets of solutions of $q = 0$ in the field has been determined; ‡ in particular, this number exceeds unity. The latter fact may be proved very simply as follows. Since the proposition is evident when $a = 0$, we may take $a = -1$, $c \neq 0$. Let $z = ty$. Then the question is whether or not $(b + ct^2)y^2 - x^2$ vanishes in the field for x and y not both zero. If not, $b + ct^2$ would be a not-square for every t. But there are only $\frac{1}{2}(p^n - 1)$ not-squares and $\frac{1}{2}(p^n + 1)$ squares t^2, including zero. Hence not every one of the $\frac{1}{2}(p^n + 1)$ values of $b + ct^2$ is a not-square.

3. Let Q be a quaternary cubic form with coefficients in the $GF[p^n]$, $p > 3$, which vanishes for no set of values in the field for x, y, z, w, except the set $x = 0, \cdots, w = 0$. The coefficient of x^3 is not zero and may be taken to be unity. By adding to x a linear function of y, z, w, we may delete the terms x^2y, x^2z, x^2w. The coefficient of x may be transformed into $dy^2 + \lambda z^2 + \mu w^2$. Then Q becomes

$$Q_1 = x^3 + x(dy^2 + \lambda z^2 + \mu w^2) + gy^3 + Ay^2z + By^2w + Cyz^2 + Dyw^2 + Eyzw + Fz^3 + Gz^2w + Hzw^2w + Rw^3.$$

Let $w = tz$. Then Q_1 becomes a ternary cubic form which may be identified with that on page 161 of volume 14 of the BULLETIN, by setting $a = 1$, $b = c = f = 0$,

$$e = \lambda + \mu t^2, \quad h = A + Bt, \quad k = C + Et + Dt^2,$$
$$l = F + Gt + Ht^2 + Rt^3.$$

Hence the relations at the top of page 165 must hold for every t. From $dh \neq 0$, we conclude that $d \neq 0$, $B = 0$. Then from the coefficient of t^3 in $eh + 3\,dl = 0$, we have $R = 0$. Hence Q_1 vanishes for $x = y = z = 0$, $w = 1$, contrary to hypothesis.

* Bachmann, Zahlentheorie, IV, p. 266, p. 553.
† *Transactions Amer. Math. Society*, vol. 10 (1909), pp. 109–122.
‡ The writer's Linear Groups, pp. 47, 48.

4. In investigating forms which represent exclusively cubes* in the $GF[p^n]$, we shall assume that p^n is of the form $3k+1$. For if p^n-1 were prime to 3, every element $\neq 0$ of the field would equal a cube in the field, and the question would be without content.

Let $ax^d+\cdots$ represent cubes exclusively. In view of the values $x=1, y=0, \cdots, a$ must be a cube. Hence we may assume that the coefficient of x^d is unity. Further d must be a multiple of 3. For, if d is prime to 3, the form represents the not-cube v^d when x equals a not-cube v, $y=0, \cdots$. *We may therefore restrict our attention to forms* $x^{3i}+\cdots$ *in the* $GF[p^n]$, $p^n=3k+1$.

5. THEOREM. *No cubic form represents cubes exclusively.*

It suffices to prove the theorem for irreducible binary cubic forms C in the $GF[p^n]$, $p^n=3k+1$. Then $C(x,1)=0$ has the roots $\rho, \rho^{p^n}, \rho^{p^{2n}}$ in the $GF[p^{3n}]$, so that

$$C=(x-\rho y)^t, \qquad t=1+p^n+p^{2n}.$$

If C represent only cubes, $C^{(p^{3n}-1)/3}=1$. Hence the power $\frac{1}{3}(p^{3n}-1)$ of $x-\rho y$ equals unity, so that $x-\rho y$ is a cube in the $GF[p^{3n}]$ for every x and y, not both zero, in the $GF[p^n]$. Any non-vanishing element μ of the latter is a cube in the $GF[p^{3n}]$. Hence

$$\mu(a-\rho)/(b-\rho) \qquad (a\neq b)$$

furnishes $p^n(p^n-1)^2$ distinct cubes in the $GF[p^{3n}]$, a number exceeding the total number $\frac{1}{3}(p^{3n}-1)$ of the cubes.

6. In the $GF[p^n]$, $p^n=3k+1$, let $S(x,y)$ be a sextic representing only cubes. Since S is the product of $S(xy^{-1},1)$ by y^6, it suffices to require that $S(x,1)$ shall represent only cubes.

First, let $p=2$, so that n is even, $n=2m$. Set

$$S=\sum_{i=0}^{6} a_i x^i \qquad (a_6=1). \tag{1}$$

Now $S^{(4^m-1)/3}=1$ since S shall represent only cubes in the $GF[4^m]$. The exponent equals $\sum_{j=0}^{m-1} 4^j$. Hence, for every x in the field,

* A non-vanishing element equal to the cube of an element of the field.

$$\prod_{j=0}^{m-1}\left(\sum_{i=0}^{6} a_i^{4^j} x^{i\cdot 4^j}\right) = 1.$$

The powers of x may be reduced by $x^{4^m} = x$. Hence, for $1 \leqq e \leqq 4^m - 1$,

$$\Sigma a_{i_0} a_{i_1}^4 a_{i_2}^{4^2} \cdots a_{i_{m-1}}^{4^{m-1}} = 0, \tag{2}$$

the sum extending over all * integers $0 \leqq i \leqq 6$ for which

$$E = i_0 + 4i_1 + 4^2 i_2 + \cdots + 4^{m-1} i_{m-1} \equiv e \pmod{4^m - 1}. \tag{3}$$

It is found to be advantageous to consider the higher values of e.

For $e = 4^m - 1$, we have $E = 2e$ (whence each $i = 6$) or $E = e$. But

$$4^m - 1 = 3 + 3 \cdot 4 + 3 \cdot 4^2 + \cdots + 3 \cdot 4^{m-1}. \tag{4}$$

Hence $E = e$ gives $i_0 \equiv 3 \pmod 4$, whence $i_0 = 3$, $i_1 = 3$; etc. Since $a_6 = 1$, (2) becomes

$$1 + a_3^{(4^m-1)/3} = 0. \tag{5}$$

For $e = 4^m - 2$, $E = 2(4^m - 1) - 1$ gives $i_0 = 5$, $i_j = 6 (j \geqq 1)$. For $E = e$, either $i_0 = 2$, $i_j = 3 (j \geqq 1)$, or $i_0 = 6$. In the latter case, $\frac{1}{4}(E - e)$ gives

$$1 + i_1 - 3 + 4(i_2 - 3) + 4^2(i_3 - 3) + \cdots + 4^{m-2}(i_{m-1} - 3) = 0.$$

Thus $i_1 \equiv 2 \pmod 4$. For $i_1 = 2$, each $i_j = 3 (j \geqq 2)$. For $i_1 = 6$,

$$1 + i_2 - 3 + 4(i_3 - 3) + \cdots + 4^{m-3}(i_{m-1} - 3) = 0.$$

Either $i_2 = 2$, $i_j = 3 (j \geqq 3)$, or $i_2 = 6$ and we proceed as before. Thus

$$a_6 + a_2 a_3^{c_1} + a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots a^{4^{m-1}}_2 a_3^{c_m} = 0 \tag{6}$$

$$[c_r = \tfrac{1}{3}(4^m - 4^r)].$$

Similarly, for † $e = 4^m - t$, $t = 3, 4, 6, 7, 8$, we get

$$a_4 + a_1 a_3^{c_1} + a_5(a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots + a_2^{4^{m-1}} a_3^{c_m}) = 0, \tag{7}$$

$$a_3 + a_0 a_3^{c_1} + a_4(a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots) = 0, \tag{8}$$

$$a_1 + a_5^{21} + a_2(a_2^4 a_3^{c_2} + a_2^{4^2} a_3^{c_3} + \cdots) + a_1^4 a_3^{c_2} + a_5^4(a_2^{4^2} a_3^{c_3} + a_2^{4^2} a_3^{c_4} + \cdots) = 0, \tag{9}$$

* Except the set in which each $i = 0$, when $e = 4^m - 1$.

† The result for $e = 4^m - 5$ is evidently the fourth power of (6).

$$(10)\qquad a_0 + a_4a_5^4 + a_1(a_2^4a_3^{c_2} + \cdots) + a_5a_1^4a_3^{c_2} + a_5^5(a_2^{4^2}a_3^{c_3} + \cdots) = 0,$$

$$(11)\qquad a_3a_5^4 + a_0(a_2^4a_3^{c_2} + \cdots) + a_4a_1^4a_3^{c_2} + a_4a_5^4(a_2^{4^2}a_3^{c_3} + \cdots) = 0.$$

For $m = 1$ the last three conditions do not occur.

First, let $a_5 = 0$. Replacing x by $x + a_4^{1/4}y$, we have $a_5 = 0$, $a_4 = 0$. By (5), a_3 is a cube in the field. By multiplying y by $a_3^{-1/3}$, we have $a_3 = 1$. Then $a_1 = 0$ by (7). By (6),

$$a_2 + a_2^4 + a_2^{4^2} + \cdots + a_2^{4^{m-1}} = 0,$$

so that $a_2 = 0$ if $m = 1$. For $m > 1$, (9) and (10) give $a_2 = 0$, $a_0 = 0$, so that the sextic vanishes for $x = 0$. Hence $m = 1$, and

$$(12)\qquad S = x^6 + x^3y^3 + y^6 = [x^3 + \sigma y^3][x^3 + (\sigma + 1)y^3],$$

where $\sigma^2 \equiv \sigma + 1 \pmod 2$ defines the $GF[4]$.

Next, for $a_5 \neq 0$, we multiply y and make $a_5 = 1$. We multiply (6) by a_3 and add the fourth power of (6). We obtain

$$1 + a_2 + a_3 = a_2a_3^{c_1+1} = a_2,$$

by (5). Hence $a_3 = 1$, so that (6) becomes

$$(6')\qquad 1 + a_2 + a_2^4 + a_2^{4^2} + \cdots + a_2^{4^{m-1}} = 0.$$

In view of the latter, (7) and (8) give

$$(7')\qquad a_1 = a_4 + a_2 + 1, \quad a_0 = 1 + a_4(a_2 + 1).$$

If $m = 1$, $a_2 = 1$ by (6′), so that *

$$S = x^6 + x^5y + a_4x^4y^2 + x^3y^3 + x^2y^4 + a_4xy^5 + y^6.$$

If $a_2^4 + a_4 = 1$, we have (13). If $a_4 = 0$, we interchange x with y and are led to the earlier case $a_5 = 0$. If $a_4 = 1$, we replace x by $x + \sigma y$, where $\sigma^2 + \sigma + 1 = 0$ and obtain a sextic S with $a_4 = 0$.

If $m > 1$, we apply (6′) to (9) and obtain

$$a_1 + 1 + a_2(a_2 + 1) + a_1^4 + a_2^4 + a_2 + 1 = 0.$$

Eliminating a_1 and a_0 by (7′), we get

$$(9')\qquad a_2^2 + a_2 = a_4 + a_4^4.$$

* In the simple case $m = 1$, we may also proceed directly with (1), which must reduce to unity (the only cube) for every x.

Condition (10) likewise leads to (9′). But (11) gives

$$a_2 + a_2^2 a_4 + a_4 + a_4^5 + a_2 a_4 = 0. \tag{11′}$$

By (9′) and (11′), we get

$$a_2 = a_4 + a_4^2.$$

By (7′), $a_1 = a_4^2 + 1$, $a_0 = a_4^3 + a_4^2 + a_4 + 1$. But $a_3 = a_5 = 1$. Hence

$$S = [x^2 + xy + (a_4 + 1)y^2]^3. \tag{13}$$

THEOREM. *In the $GF[2^n]$, n even, every binary sextic which represents only cubes is formally a perfect cube, except for the case $n = 2$, when there is an additional type* (12), *the product of two irreducible cubic forms.*

7. For $p^n = 3k + 1$, $p > 2$, we may remove the term x^5y from the binary sextic S. Hence it suffices to investigate

$$S = z^6 + bz^4 + cz^3 + dz^2 + ez + f \quad (f = \text{cube}). \tag{14}$$

For $p^n = 7$, the cubes are ± 1. We require that $S^2 \equiv 1$ for every z making $z^6 \equiv 1 \pmod 7$. First, let $f = -1$. Then

$$(bz^4 + cz^3 + dz^2 + ez)^2 \equiv 1$$

requires that

$$be + cd \equiv 0, \quad d^2 + 2ce \equiv 0, \quad de \equiv 0, \quad b^2 + e^2 \equiv 0,$$
$$bc \equiv 0, \quad c^2 + 2bd \equiv 1.$$

Hence $S = z^6 \pm z^3 - 1$. Multiplying z by ± 1, we get

$$z^6 + z^3 - 1 = (z^2 - z + 3)(z^2 - 2z - 2)(z^2 + 3z - 1). \tag{15}$$

Next, let $f = 1$. Then $(bz^4 + \cdots + 2)^2 = 1$ gives

$$be + cd \equiv d^2 + 2ce + 4b \equiv 2c + de \equiv b^2 + e^2 + 4d$$
$$\equiv bc + 2e \equiv c^2 + 2bd + 3 \equiv 0.$$

If $e \equiv 0$, then $c \equiv 0$, $d \equiv -2b^2$, $b^3 \equiv -1$. Multiplying z by $3b^{-1}$, we have $b = 3$. Hence

$$S = (z^2 + 1)^3. \tag{16}$$

If $e \neq 0$ we multiply y by e and have $e = 1$. Then the first three conditions give

$$b \equiv 2c^2, \quad c^2 \equiv c, \quad d \equiv 5c.$$

The fifth gives $c^3 \equiv -1$, in contradiction with $c^2 \equiv c$.

Any sextic $ax^6 + \cdots$ which represents only cubes, modulo 7, cannot equal -1 for every x and y, not both zero, and hence must take the value $+1$. Hence it can be transformed into $x^6 + \cdots$.

THEOREM. *Every binary sextic which represents only cubes, modulo 7, can be transformed into*

$$(x^2 + y^2)^3 \quad or \quad x^6 + x^3y^3 - y^6.$$

8. Let $\lambda = z^6 + bz^4 + dz^2 + f$, $\mu = cz^3 + ez$. Then $\lambda + \mu$ and $\lambda - \mu$ must be cubes for every z. For $p^n = 13$, $(\lambda \pm \mu)^4 \equiv 1$, whence

$$(17) \quad \lambda \equiv 0, \mu^4 \equiv 1\,; \ \mu \equiv 0, \lambda^4 \equiv 1\,; \ \lambda^2 + \mu^2 \equiv 0, \lambda^4 \equiv \mu^4 \equiv 3 \pmod{13}.$$

Hence for every z,

$$(18) \qquad \mu(\mu^4 - 1)(\mu^4 - 3) \equiv 0 \pmod{13}.$$

If $e \equiv 0$, $\mu = cz^3$, and (18) gives

$$c(c^4 - 1)(c^4 - 3) \equiv 0 \pmod{13}.$$

If $c^4 \equiv 1$, then $\mu^4 \equiv 1$ for $z \not\equiv 0$, so that, by (17), $\lambda \equiv 0$ for every $z \not\equiv 0$, contrary to the degree of λ. If $c^4 \equiv 3$, then $\mu^4 \equiv 3$, $\lambda^2 + \mu^2 \equiv 0$ for $z \not\equiv 0$; but the resulting conditions give $b \equiv d \equiv 0$, $c^2 \equiv \pm 3$, contrary to $c^4 \equiv 3$.

For the troublesome case $c = e = 0$, we set

$$\lambda = z^6 + dz^2, \quad \mu = bz^4 + f,$$

so that $S = \lambda + \mu$ is a cube. Replacing z by $5z$, we see that $-\lambda + \mu$ is a cube for every z. Hence one of the three cases (17) must hold. Now $\mu(\mu^4 - 3)$ involves only powers of z^4 and hence, for $z \not\equiv 0$, equals a function of degree 8. Thus at least four values $\not\equiv 0$ of z must give rise to the first case (17). Hence $\lambda = 0$ has four roots $z \not\equiv 0$, so that $d^3 = -1$. Multiplying y by d^{-1}, we may set $d = -1$. Thus $\lambda = z^2(z^4 - 1)$. When $\lambda \not\equiv 0$, we have $z^4 = 3$ or 9, viz.,

$$z^4 = 3, \quad \lambda = 2z^2, \quad \lambda^2 = -1, \quad \lambda^4 = 1\,;$$

$$z^4 = 9, \quad \lambda = 8z^2, \quad \lambda^2 = 4, \quad \lambda^4 = 3,$$

thus corresponding to the respective cases (17). Thus if $z^4 = 3$, then $\mu = 0$, whence $3b = -f$. For $z^4 = 9$, then $\mu^2 + 4 = 0$, $f^2 = -1$. Hence $S = (z^2 \pm 2)^3$.

Finally, let $e \neq 0$. Since every integer is a fifth power modulo 13, we may multiply y and make $e = 1$. Then (18) holds for $\mu = cz^3 + z$. For $z \neq 0$,

$$\mu^8 - 4\mu^4 + 3 = (8c^7 - 3c^5 + 8c)z^{10} + (2c^6 + 2c^2 + 1)z^8$$
$$+ (4c^5 - 3c)z^6 + (5c^4 - 4)z^4 + 4c^3z^2 + c^8 - 4c^4 + 2c^2 + 3.$$

In its product by $cz^2 + 1$, the coefficients of z^4 and z^6 are

$$9c^4 - 4, \quad 9c^5 - 7c,$$

which do not both vanish modulo 13.

Hence the sextics which represent only cubes modulo 13 have been reduced to the product of $(x^2 \mp 2y^2)^3$ by a cube l^6. But $l(x^2 \pm 2y^2)$ can be transformed into $x^2 + 2y^2$.

THEOREM. *Every sextic which represents only cubes modulo* 13 *is a perfect cube which may be transformed into* $(x^2 + 2y^2)^3$.

9. For the next two values 19 and 25 of $p^n = 3k + 1$, I find that any binary sextic S which represents only cubes equals the cube of a quadratic form. For $p^n = 5^2$, this result follows readily by requiring that S^2 shall equal its fifth power; by the coefficients of z^{21}, z^{16}, z^{11}, z^4, we get $S = (z^2 + 2b)^3$.

For $p^n = 11^2$, S^4 must equal its 11th power. By the coefficients of z^{111}, z^{100}, z^{89}, z^{78}, we get $S = (z^2 + 4b)^3$.

It seems probable that, for $p^n = 3k + 1 > 7$, every binary sextic which represents only cubes is formally a perfect cube. Although certain general considerations point to the validity of this conjecture, I have not effected a complete proof.

10. A sextic on three or more variables cannot represent exclusively cubes in a field of order $p^n = 3k + 1$ for which the binary sextics representing only cubes are formally perfect cubes. It suffices to prove the theorem for ternary sextics T. For * $p > 2$, we may delete the terms x^5y, x^5z, y^5z. Then the terms free of z are $(x^2 - vy^2)^3$, where v is a not-square. When z is multiplied by a suitable constant, the terms free of y are $(x^2 - vz^2)^3$. Then the terms free of x are $-v^3(y^2 + \epsilon z^2)^3$, where $\epsilon^3 = 1$. Since ϵ equals the square of ϵ^2, and $y^2 + \epsilon z^2$ is irreducible, -1 must be a not-square. Hence we may set $v = -1$. Hence

$$T = x^6 + y^6 + z^6$$
$$+ 3(x^4y^2 + x^2y^4 + x^4z^2 + x^2z^4 + \epsilon y^4z^2 + \epsilon^2y^2z^4) + xyz\phi, \tag{19}$$

* By a different argument, I have proved that there is no tenary sextic in the $GF[2^{2m}]$ which represents only cubes.

$$\phi = Ax^3 + By^3 + Cz^3 + Dx^2y + Ex^2z + Fxy^2 + Gxz^2 + Ky^2z + Iyz^2 + Mxyz. \tag{20}$$

Set $x = \lambda y$. Then, for λ arbitrary in the field,

$$f = (\lambda^2 + 1)^3y^6 + ry^5z + sy^4z^2 + (G\lambda^2 + L\lambda)y^3z^3 + ty^2z^4 + z^6$$

must be formally a perfect cube in x, y. Here

$$r = A\lambda^4 + D\lambda^3 + F\lambda^2 + B\lambda,$$
$$s = 3\lambda^4 + E\lambda^3 + M\lambda^2 + K\lambda + 3\epsilon,$$
$$t = 3\lambda^2 + C\lambda + 3\epsilon^2.$$

Hence $f = [(\lambda^2 + 1)y^2 + \omega z^2]^3$, where $\omega^3 = 1$. Thus

$$r = 0, \quad s = 3\omega(\lambda^2 + 1)^2, \quad t = 3\omega^2(\lambda^2 + 1), \quad G\lambda^2 + L\lambda = 0$$

for every λ in the given field of order > 4. Hence

$$A = B = C = D = E = F = G = K = L = 0, \quad M = 6,$$
$$\omega = \epsilon = 1, \quad T = (x^2 + y^2 + z^2)^3.$$

By § 2, T vanishes for x, y, z, not all zero, in the field.

11. It remains to investigate the ternary sextics T which represent only cubes in the $GF[7]$. The terms free of x in T will be denoted by B_{yz}, those free of y by B_{xz}, etc. By § 7, each B is a perfect cube or a product π of three quadratic forms. Suppose that one of the B's, say B_{xz}, is of the type π. Replacing x by $x + ry + sz$, we may delete the terms x^5y and x^5z; then B_{xz} is still of the type π, and B_{xy} is of its original type. Hence we may set

$$B_{xz} = z^6 + x^3z^3 - z^6.$$

Let T become T' when y is replaced by $y + \rho z$. The coefficient c of z^6 in B'_{xz} is the value of B_{yz} for $y = \rho$, $z = 1$, and hence is a sextic in ρ with a non-vanishing coefficient for ρ^6; thus $c \neq -1$ for some value of ρ. Since B'_{xy} is not of type π, it is a perfect cube. Also, $B'_{xy} = B_{xy}$.

Thus one of the B's may be assumed to be a perfect cube. With B_{xy} a cube, the same argument shows that we may make also B_{xz} a perfect cube, and that we may set

$$B_{xz} = (x^2 + z^2)^3, \quad B_{xy} = (x^2 + y^2)^3.$$

Employing the abbreviation ϕ, given by (20), we have

$$T=x^6+y^6+z^6+3x^4y^2+3x^4z^2+3x^2y^4+3x^2z^4+\sum_{i=1}^{5} c_i y^{6-i}z^i+xyz\phi.$$

Set $y=\lambda z$. Then T becomes

$$t=x^6+r_2x^4z^2+r_3x^3z^3+\cdot+r_6z^6,$$

$r_2=3\lambda^2+A\lambda+3$, $r_3=D\lambda^2+E\lambda$, $r_4=3\lambda^4+F\lambda^3+M\lambda^2+G\lambda+3$,

$r_5=B\lambda^4+K\lambda^3+L\lambda^2+C\lambda$, $r_6=\lambda^6+c_1\lambda^5+\cdots+c_5\lambda+1$.

Now $\tau=x^6\pm x^3z^3-z^6$, viz., is of type π, if and only if

$$r_2\equiv r_4\equiv r_5\equiv 0,\quad r_3^2\equiv 1,\quad r_6\equiv -1 \pmod 7; \tag{21}$$

while τ is a perfect cube if and only if

$$r_3\equiv r_5\equiv 0,\quad r_4\equiv 5r_2^2,\quad r_6\equiv 6r_2^3 \pmod 7. \tag{22}$$

Since $r_2r_3\equiv 0$ for every λ, $D\equiv E\equiv 0$. Hence (21) is excluded, so that (22) must hold for every λ. We may therefore remove the term y^5z from T and proceed as in § 10. Or we may proceed with (22) and show that

$$T\equiv(x^3+y^3+z^3-2Axyz)^3.$$

12. The theorem that there exists no sextic on three or more variables which represents only cubes in a field of order $p^n=3k+1$ has now been established for $p^n<31$, $p^n=2^n$, and $p^n=11^2$. Its truth for all values of p^n has been proved, subject to the validity of the conjectured theorem of § 9 on binary sextics.

THE UNIVERSITY OF CHICAGO,
October, 1908.

NOTE ON LÜROTH'S TYPE OF PLANE QUARTIC CURVES.

BY PROFESSOR H. S. WHITE AND MISS KATE G. MILLER.

(Read before the American Mathematical Society, September 6, 1907.)

ONE of the stock examples of the fallacy of constant counting is the equation of a plane quartic, whose fourteen constants equal in number those apparent in the sum of five fourth powers of linear expressions

$$x_1^4+x_2^4+x_3^4+x_4^4+x_5^4.$$

Clebsch showed why this is not a normal form for all ternary quartics; it has by virtue of its form an infinite system of linear transformations into itself, and so contains actually but thirteen constants. The curve has an apolar, *i. e.*, a triply apolar conic, and the sides of its polar quinquilaterals form a simple quintic involution of tangents to this conic. This special type of quartic is called the Clebschian.

Lüroth's quartic, closely related to this, is an equally alluring source of fallacy. It contains inscribed complete quinquilaterals, —all ten vertices lying on the curve. At first sight one would say that every quartic might be of this type. For we may take three variable points, A, B, C on the curve, join them by straight lines, and so have a triangle with three degrees of freedom. Each side cuts the curve in two more points, say L_1 and L_2 on BC, M_1 and M_2 on CA, and N_1, N_2 on the third side. One condition is imposed on the variable points by requiring three points L_1, M_1, N_1, to be collinear; let their line meet the quartic again in P. Finally we should expect two more conditions to be imposed by requiring the four points L_2, M_2, N_2, and P to be collinear; three conditions in all upon the three variable points A, B, and C. If so, the construction might be possible for a finite number of positions, and for each position the five lines of the diagram would constitute an inscribed complete quinquilateral. Unfortunately for the argument, one such inscribed quinquilateral in a quartic curve proves the existence of an infinite number of others instead of a finite number, hence our hasty inference that this type is the most general quartic is nugatory. Lüroth's quartic and the Clebschian are related by the fact that all the inscribed quinquilaterals of the one are the polar quinquilaterals of the other.

Lüroth's quartic is evidently a projective covariant curve of the Clebschian, and has been studied as such. If a certain invariant B (the catalecticant) of the ternary quartic c_x^4 or c vanishes, then c is of the Clebsch type, and its covariant S gives the equation $S = 0$ of the corresponding Lüroth quartic

$$S = (cc'c'')(cc'c''')(cc''c''')(c'c''c''')c_x c'_x c''_x c'''_x = 0.$$

The direct and unique connection between the two is thus completely exhibited when the Clebschian is given as the starting point. From it both the Lürothian and the related conic are found as rational covariants,* and the condition which

*See statements and references by Coble in *Transactions Amer. Math. Society*, vol. 4 (1903), pp. 65–85.

must be satisfied is expressed by a rational invariant equation. But suppose the Lürothian equation to be given, $l_x^4 = 0$. Could we then find rationally in its coefficients the Clebschian, the conic, and the characteristic invariant condition for the Lürothian type? This has not been announced, yet apparently it should be possible. We wish here to outline a possible answer, although the result is not so simple as to be entirely satisfactory.

Let the Lürothian, Clebschian, and conic be represented by the equations

$$l_x^4 = 0, \quad c_x^4 = 0, \quad \text{and} \quad u_a^2 = 0,$$

respectively. These are connected by a relation that seems not to have been employed hitherto. It is well known that tangents to the conic are cut by the two quartics in apolar quartettes of points. But it is further true that *these tangents are the only lines whose intersections with the quartics are mutually apolar.* The envelope of such lines is of the fourth class,

$$(clu)^4 = 0,$$

and if the conic forms part of this envelope, a second conic must form the complementary part. That this complementary part is the same conic repeated can be verified easily from the normal forms of the equations

$$l_x^4 = \sum_1^5 (x_2 x_3 x_4 x_5) \quad \text{and} \quad c_x^4 = \sum_1^5 x_1^4.$$

We have therefore the formal equation, identical in the u's,

$$(lcu)^4 \equiv (u_a^2)^2. \tag{1}$$

The triple apolarity of conic and Clebschian is expressed by the identity in the variables x_1, x_2, x_3,

$$c_a^2 c_x^2 \equiv 0. \tag{2}$$

The identity (1) yields 15 equations linear in the quantities $c_{ijk} (= c_1^i c_2^j c_3^k)$; from these we find the Clebschian explicitly as a fifteen-rowed matrix bordered with one column of the second degree in the coefficients of u_a^2, and one row quartic in variables x_1, x_2, x_3. Call the determinants $|L|$, and the bordered matrix

$$\begin{vmatrix} L & (\alpha)^2 \\ (x)^4 & 0 \end{vmatrix},$$

and we write

$$(3) \qquad |L| \cdot c_x^4 \equiv \begin{vmatrix} L & (\alpha)^2 \\ (x)^4 & 0 \end{vmatrix}.$$

Hence, applying condition (2), we find what may be symbolized thus

$$(4) \qquad \begin{vmatrix} L & (a, \alpha) \\ (\alpha, x^2) & 0 \end{vmatrix} \equiv 0 \text{ for all values of the } x_1, x_2, x_3.$$

This contains six cubic equations for the six coefficients of the conic. *If these can be satisfied simultaneously, the quartic l_x^4 is of Lüroth's type; equation (3) gives the Clebschian in terms of the Lürothian and the conic; and the common solution*

$$(\alpha_{11}, \alpha_{12}, \cdots, \alpha_{33})$$

of the six equations (4) is the conic envelope.

We may state the relation of things more concisely by using the terminology of space of five dimensions, treating the coefficients α_{ik} of the conic as homogeneous coordinates in such a space. Replace the powers and products of x_1, x_2, x_3 in equation (4) by a second set of coordinates β_{ik} cogredient with the α_{ik}; then equation (4) is that of the first (cubic) polar of a point β in the five-space, with respect to a quartic locus or four-spread in that space, whose equation may be denoted by

$$(5) \qquad \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{vmatrix} = 0,$$

since

$$\begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, x^2) & 0 \end{vmatrix}_{x_i x_k = \beta_{ik}} \equiv \tfrac{1}{2} \Sigma \left(\beta_{ik} \frac{\partial}{\partial \alpha_{ik}} \right) \cdot \begin{vmatrix} L & (\alpha, \alpha) \\ (\alpha, \alpha) & 0 \end{vmatrix}.$$

This first polar equated to zero, identically in the β_{ik}, is the requirement that the quartic four-spread (5) shall have an ordinary double point or node, viz., at the point (α). Therefore, *the problem of finding the invariant condition satisfied by the Lürothian is equivalent to the problem of finding the discriminant of a certain quartic (5) homogeneous in six variables. If that discriminant vanishes, the determination of the covariant conic α is effected by determining the coordinates of the node of that quartic four-spread; and from this conic and the Lürothian the Clebschian c_x^4 is then found uniquely by equation (3).*

That this solution shall not be illusory, of course the determinant $|L|$ must not vanish. For a particular quartic, one containing only even powers of x_1, x_2, x_3, this test is made very easily; and it is of interest to record that then this determinant $|L|$ factors into two rational expressions of degrees 6 and 9 respectively, neither of them containing Salmon's invariant B as a factor. This shows that $|L|$ is not zero for a quartic L of the Clebschian type. But also for the Lüroth type of quartic it does not vanish. This we have proved with some labor by taking l_x^4 in the canonical form

$$l_x^4 = Ax_2x_3x_4x_5 + Bx_1x_3x_4x_5 + \cdots + Ex_1x_2x_3x_4,$$

with the two linear relations

$$x_4 = \alpha_1x_1 + \alpha_2x_2 + \alpha_3x_3, \quad x_5 = \beta_1x_1 + \beta_2x_2 + \beta_3x_3,$$

and computing a sample term of $|L|$. We selected the one containing the quantities C and β_3 to the highest possible powers, namely the term in

$$A^2B^2C^{11}\alpha_1^7\alpha_2^5\alpha_3^3\beta_2^2\beta_3^{13}.$$

Such a term can occur in only 21 terms of the expanded determinant $|L|$, and carefully repeated computation gives its coefficient as -8. While we should prefer some simpler verification of this point, yet we regard this trial as sufficient.

Owing to the high degree of the discriminant of a senary quartic, it seems at least likely that for a quartic of the particular type $\left|\begin{smallmatrix} L & (a_1, a) \\ (a, a) & 0 \end{smallmatrix}\right|$ the discriminant might prove to be an exact power of some invariant of lower order in the given coefficients l_{ik}. Such a reduction we have proved to exist, so far, only in certain analogous forms in fewer variables.

When the Lürothian is regarded as derived from a quintic involution among the tangents of a conic, F. Meyer has shown precisely how to determine the (4, 4) correspondence among those tangents that represents the Clebschian. This mode of determination he extends to forms of any even order. An involutional sextic curve is always uniquely associated with another sextic which is apolar to the septilaterals inscribed in the former. The relation between such a pair of sextics (octics, etc.) in ternary variables can be expressed by an obvious generalization of the identities (3) and (4). For example if $l_x^6 = 0$ and $c_x^6 = 0$ are

two such sextics, and $u_a^2 = 0$ the conic carrying the related involution, then the envelope

$$(clu)^6 = 0$$

is the triply counted conic: $(clu)^6 \equiv \rho \cdot (u_a^2)^3$. So for the similarly related pairs of higher order.

VASSAR COLLEGE.

CANTOR'S HISTORY OF MATHEMATICS.

Vorlesungen über Geschichte der Mathematik. By MORITZ CANTOR. Vierter Band. Von 1759–1799. Leipzig, Teubner, 1908. 8vo. vi + 1113 pp. 32 Marks.

IT is a long period that Moritz Cantor looks back upon to the time when he made his first noteworthy contribution to the history of mathematics. He was little more than a boy when he published his Inaugural-Dissertation "Ueber ein weniger gebräuchliches Coördinaten-System" in 1851,—only twenty-two years old; and it was only five years later that he entered upon his life work as historian of mathematics, by the publication of his paper "Ueber die Einführung unserer gegenwärtigen Ziffern in Europa." He was thirty-four when his first noteworthy treatise on the history of mathematics appeared, the "Mathematische Beiträge zum Kulturleben der Völker" (Halle, 1863). And next August he will be eighty years old, with over a dozen books to his credit, with hundreds of memoirs, with a notable record as editor of the *Zeitschrift für Mathematik und Physik* and the *Abhandlungen zur Geschichte der Mathematik*, and with academic honors and governmental recognition commensurate with the great work that he has accomplished.

For a man of nearly eighty to undertake a work of the magnitude of this fourth volume of the Geschichte would seem foolhardy, were he not such a man as Professor Cantor. When the third volume was completed in 1898 it seemed proper that it should be called the last, as was the case, and under all the circumstances any man might have been well content to say his *nunc dimittis*. That Professor Cantor was not thus content is a cause of gratification to all who are interested in the history of mathematics, and a worthy lesson to all who feel that threescore years mark the bounds of man's working life.

The present volume is not the work of Cantor himself in the

same way as the earlier part of the history, although it was planned and edited by him and contains one chapter from his own pen. He has brought to his assistance several of the best men in Germany and in other countries, each an expert in the line upon which he has written. The following are the chapters and authors: Histories of mathematics, by S. Günther; Arithmetic, algebra, and theory of numbers, by F. Cajori; The combination theory, doctrine of probabilities, series, and imaginaries, by E. Netto; Elementary geometry, by V. Bobynin; Trigonometry, polygonometry, and tables, by A. von Braunmühl; Analytic geometry of the plane and of space, by V. Kommerell; Perspective and descriptive geometry, by G. Loria; Infinitesimal calculus, by G. Vivanti; Total and partial differential equations, sums and differences, and the calculus of variations, by C. R. Wallner; Survey of the period from 1758 to 1799, by M. Cantor.

Professor Günther's article relates to the histories of mathematics, including monographs and editions of the classics, published during the period in question. The chapter is particularly valuable for the reason that we have no worthy bibliography of the subject in any period preceding the founding of *Bibliotheca Mathematica* and the *Jahrbuch über die Fortschritte der Mathematik*, and this particular period is practically the one in which serious historical investigation begins. A rather careful examination of this chapter fails to show the omission of any names of importance, and the chapter will stand as a worthy bibliography of the period. One or two minor misprints have been noticed, but they are not worth mention at this time.

It is a matter of gratification to Americans that Professor Cajori should appear as one of the collaborators in this work. To him was assigned the subject of arithmetic, theory of equations, and theory of numbers. The period has not been carefully studied before, and it is apparent even on first reading that Professor Cajori's best work has been done here. He has gone to a sufficient number of original sources to trace the development at first hand, and his secondary authorities are standards. Not the least valuable is the brief but comprehensive sketch of early American textbooks. Some points are open to criticism, and naturally so when secondary authorities are taken. For example, Trenchant (1566) commonly used miliar for 10^9, and miliar de miliars for 10^{12}. As to the date of Ruffini's Teoria generale delle equazioni, upon which Professor Cajori raises a

question, it is not improbable that the authorities giving 1798 (and the excellent Elogio by Giuseppe Bianchi might well have been included) are quite as correct as those giving 1799. In my own copy the title pages of both volumes originally read MDCCXCVIII, but in both cases an additional I has been added, perhaps by a type held in the hand. It is probable that some questions may arise with respect to a few of the other dates, particularly of memoirs, for so much material has been used that slight errors are to be expected.

Netto's contribution begins with the so-called kombinatorische Schule, founded by Hindenburg whose first publication on the subject was made at the age of thirty-nine, although Æpinus, van Schwinden, and Euler had already made some use of the idea in connection with the binomial series, the last named having suggested the common German symbolism for the coefficients and having stated two well-known problems in the theory. In the doctrine of probabilities the author shows our great indebtedness to D'Alembert, who, building upon the work of Jakob Bernoulli, contributed much more to the theory than has commonly been supposed. The work of Laplace has generally been appreciated, probably because of the extensive use of least squares, and possibly because of the interest immediately aroused by his study of vital statistics, and Professor Netto has given an excellent resumé of this phase of Laplace's work. In the study of series the beginning is made with the work of Landen. This is followed by the contributions of Euler, Daniel Bernoulli, Pfaff, and other continental writers, and also of the too generally neglected English scholars, Vince, Waring, and Hellins. In the subject of imaginaries Professor Netto also gives much more credit to D'Alembert than has generally been the case. He shows that Euler first used i for $\sqrt{-1}$, in his Institutiones calculi integralis appearing the words, "formulam $\sqrt{-1}$ littera i in posteriorem designabo." As to the graphic representation of complex numbers he gives the credit, of course, to Wessel.

Professor Bobynin begins his chapter on elementary geometry by a discussion of textbooks, showing very effectively by table the great advance of Germany and France, in this phase of education, over the other European countries, and especially over England. Here the influence of D'Alembert again stands out prominently, and Bobynin's appreciation following immediately that of Netto suggests that in spite of the praise that

D'Alembert has generally received he is entitled to still more recognition. It is, however, to Legendre that the author assigns the greatest honor, and with good reason, and the essential features of his revision of the Euclid sequence are carefully set forth. The second topic in this chapter is on practical geometry or surveying, and here the noteworthy advance naturally centers around the establishing of the basal unit of the metric system. We are already in possession of an excellent treatise on this subject in the work of Bigourdan, but Professor Bobynin has approached it more definitely from the geodetic standpoint. The third topic relates to specific propositions, and particularly to Lexell's contributions. The fourth topic considers the theory of parallels, necessarily closing before the modern Lobachevsky-Bolyai developments.

The contribution of the late Professor von Braunmühl is along the line of his great life work, the history of trigonometry. It begins with the period of Euler and his contemporaries, the same as is covered, often more fully, in the author's history, chapters 4 and 5. The date of Blake's birth, on the second page, is wrong, by the way; it should be 1708 instead of 1718. The period is one of peculiar interest, covering a good part of Euler's most prolific years, but the treatment is necessarily more condensed than in von Braunmühl's larger work. The second topic relates to efforts to improve the teaching of trigonometry, especially as to its foundation principles. The third part relates to tetragonometry, polygonometry, and polyhedrometry, and the fourth part to tables, cyclometry, and trigonometric series. In several places the latter part of von Braunmühl's contribution is more complete than his history.

Professor Kommerell's chapter on analytic geometry of the plane and of space covers a field that has been less worked than that of the preceding chapter or two. He begins with some general notes on conics and then takes up higher plane curves, proceeding from Waring's Proprietates algebraicarum curvarum, of which he gives a careful resumé. The period was one of great activity in this line of work, and some fifty pages are assigned to the subject. Space curves and surfaces are next discussed, beginning again with Waring, this part of the treatment also covering about fifty pages.

Professor Loria's article is divided into four topics. The first treats of perspective, but unlike the other contributors the author has found it necessary to begin with the middle ages on

account of omissions in the earlier volumes of the history. He brings out in a clear if not new light the indebtedness of mathematics to the graphic arts in the fifteenth and sixteenth centuries, and makes further mention of the work of Desargues in the seventeenth century. His second topic is The golden period of theoretic perspective, in which he begins with the work of s'Gravesande and properly pays high tribute to the genius of Lambert. The third topic touches briefly upon the immediate predecessors of Monge, and the fourth devotes some fifteen pages to the great founder of descriptive geometry.

The treatment of the infinitesimal calculus by Professor Vivanti opens with a consideration of the foundations of the subject, carrying the work on from where it was interrupted in volume III. The second topic deals with the textbooks of the period. The third is subdivided, the first part treating of processes of differentiation, in which the work of Pfaff is most prominent, and the second treating of integration, a subject particularly fertile of results in this period. The fourth topic relates to definite integrals, beginning and necessarily chiefly concerning itself with the contributions of Euler. The fifth topic relates to the analytic applications of the infinitesimal calculus, beginning with the English contributors to the theory of maxima and minima, and including the development of indeterminate forms and series. Professor Vivanti's contribution closes with an exhaustive discussion of transcendents and elliptic integrals, some eighty pages in all, beginning with Vandermonde's treatment of the theory of factorials, but chiefly devoted to the theory of elliptic integrals. These functions, while first suggested somewhat earlier, had their greatest development in this period, and chiefly at the hands of Euler. While the subject has already been well treated by Enneper and by Bellocchi, the author has examined at first hand so much source material that his own work must be used to supplement that of his two predecessors.

Professor C. R. Wallner's contribution to the history of total and partial differential equations, differences and sums, and the theory of variations, is one of the most exhaustive in the work. While the theory of differential equations had already been established before 1750, their solution at the outset had been looked upon as means to an end rather than an object in itself. It was during the latter half of the eighteenth century, the period covered by this volume, that the theory came to its

complete independence, and chiefly through the influence of Euler, D'Alembert, Danie Bernoulli, Lagrange, Laplace, and Legendre.

Professor Cantor's own contribution is largely bibliographical, consisting of a list of the most important works mentioned by his collaborators. He makes the same assignment of date to Ruffini's first work on equations of the fifth degree as made by Professor Cajori. It is evident that the printing of the work began in 1798 and was completed in 1799.

It is too early to enter into critical details of such a work. That numerous errors will be found is certain, as witness the partial list given by Müller, and the fact that the *Bibliotheca Mathematica* gives every month a list of corrections extending back to the first edition of the first volume, published nearly thirty years ago. On the other hand it will be a long time before any one will attempt to treat so exhaustively this remarkable period in which the genius of Euler, D'Alembert, Lagrange, Laplace, and Legendre showed at its best, and in which Gauss was beginning the labors that placed his name among the leaders of modern times. DAVID EUGENE SMITH.

SHORTER NOTICES.

High School Algebra. By H. E. SLAUGHT and N. J. LENNES. *Elementary Course*, 1907, vii + 297 pp., $1.00; *Advanced Course*, 1908, vii + 194 pp. $0.65. Boston, Allyn and Bacon.

THE common weakness of our college students in elementary algebra shows a great need for improvement in the teaching of this subject. The appearance of these admirable texts, constructed after a new model, marks a distinct advance in the teaching of algebra in our high schools. Recent discussions have shown that it is quite generally agreed that high school algebra should be divided into a first course of elementary algebra during the first year, and a second course of review and advanced algebra during one half of the third or fourth year, preceded by one year of plane geometry. The authors have met this demand of teachers by dividing their text into two parts, and in doing so have succeeded in presenting the subject of high school algebra in the most concrete and teachable form we have yet seen.

As stated in the preface, "the main purpose of the Elementary Course is the solution of problems rather than the construction of a purely theoretical doctrine as an end in itself." Throughout the first course the method is inductive. The interests, ability, and needs of the first year high school students seem to demand this selection of the inductive method. In writing elementary texts, mature mathematicians too often let their delight in the subject as an elegant deductive system obscure the ability of the student at this particular stage in his development. The teachableness of this book is greatly increased by the fact that the authors never lose sight of the standpoint of the *pupil*.

The illustrative problem is used throughout the book to show the need of new processes and to secure real insight into the principles involved. This enables the authors to build upon the arithmetic knowledge and experience of the pupils, to proceed from the known to the unknown, and to make the work more concrete. Illustrative problems are made to take the place of long abstract descriptions and explanations. New notions and processes thus first made evident by illustrative exercises are always followed up by a careful and precise statement of the *principle* involved. By this method eighteen principles used in the Elementary Course are introduced. These eighteen principles stand out prominently as a body of facts which each student must know and be able to use. It is somewhat of a surprise that the authors have been able to reduce the subject of elementary algebra to such a small number of fundamental principles.

New ideas are introduced one at a time and only when needed in the solution of problems. This, with a desire to secure a maximum of concrete matter in the early parts of the book, has led to a slightly new order of topics. The most noticeable change from the conventional order is in the position of factoring, long division, and literal fractions. Factoring follows simultaneous linear equations, where first needed as an introduction to the quadratic. Long division comes quite late in the course because it is first needed in the square root process required in the general solution of the quadratic. Literal fractions come in the last chapter because the knowledge of arithmetic fractions and the use of simple operations are sufficient for earlier problems. This new order of topics is justified by the gain in more early concrete work and in placing the

complicated work later in the course. Of course such an order of topics requires greater care in the grading of problems, and this has been exercised.

Since the chief purpose is the solution of problems, the equation as an instrument for this purpose is early introduced and occupies a prominent place throughout the book. The analogy drawn between a balance and an equation is good. Equations are solved for many different letters, which is excellent preparation for later work in physics. The real test for a solution, namely that it satisfies the equation, is properly emphasized by many exercises in checking. The form of arrangement in the illustrative problems requires a justification of each operation by an explicit reference to some one of the eighteen fundamental principles. This tends to overcome the common fault of blind juggling of symbols and lays an excellent foundation for the deductive proofs of the Advanced Course. On nearly every page appear evidences of a purpose to develop ability to translate English into the forms of algebra; many parallel statements of the same fact in English and algebra are found; formulas are restated in words; numerous exercises in translation are given.

Negative numbers are well introduced and treated by means of concrete examples; emphasis is placed on the interpretation of the sign of the result of a problem. The term "less than zero" which confuses beginners is avoided. The term "absolute value" is introduced in this connection.

The justification for the introduction of graphs into algebra is their usefulness in illustrating the principles involved in simultaneous equations. The authors introduce graphs just *before* considering linear equations and then *use* them in their treatment of simultaneous equations throughout both books. The usefulness of graphs is made so evident that they become an integral part of the subject, instead of seeming to be an extraneous topic tacked on to algebra.

The emphasis placed upon elimination by substitution should be noted. Real insight into what is happening comes through the process of substitution. Elimination by addition and subtraction is given second place. Elimination by comparison is omitted in the Elementary Course but is found in the Advanced Course.

Factoring is introduced on page 172 — about the middle of the course. Complicated cases are omitted in the first course. The treatment of the type $ax^2 + bx + c$ is especially good. In

each set of exercises for factoring there are some expressions which are not factorable by the case under consideration; this has the advantage of requiring the pupil to study the *form* of each expression.

The treatment of the quadratic illustrates the authors' use of the spiral method. The quadratic first appears on page 197 as an application of factoring; on page 240 as an application of square root; quadratic and linear simultaneous equations come on page 246; while the complete theory of the quadratic comes in the Advanced Course. This spiral method has the advantage of going from the simple to the complex and gives a good review of a topic by repeated considerations, each time in a more complicated form.

Literal fractions appear in the last chapter. It is surprising how many problems can be solved without the use of literal fractions. In the treatment of complex fractions the shorter method of multiplying both terms of the complex fraction by the least common multiple of all the minor denominators is used. This method is too frequently neglected.

Numerous definitions appear in the Elementary Course, but a tendency to avoid or postpone the introduction of technical terms is noted. The authors seem to feel that by repeating the description of technical terms, such as *transpose* or *equivalent equations*, each time they are needed a greater insight into real processes is obtained.

The reviewer was impressed by the number of concrete problems; in the first 150 pages there are about 65 pages of concrete problems and 25 pages of abstract exercises. The problems contain interesting information and are of such a nature as constantly to raise the question whether the result is reasonable. The desirable habit of appealing to one's common sense to justify results is thus strengthened. The inductive principle is even carried into the problems. General or literal problems which result in formulas are approached through a series of numerical problems. The close connection of algebra and arithmetic is brought out by many problems in Arabic numbers; nearly every principle is illustrated by Arabic numbers.

The Advanced Course presupposes a knowledge of the principles of elementary algebra, considerable training in deductive reasoning obtained in the preceding year of plane geometry, and greater maturity in the pupils; hence its point of view is entirely different from that of the Elementary Course. Its

method is deductive, and instead of principles to be illustrated the algebraic facts are put in the form of theorems to be proved. This new point of view gives a fine review of the Elementary Course without degenerating into an uninteresting rehash of old work.

The first chapter is given to a careful statement of the axioms used. The nine axioms cover the assumptions made concerning the uniqueness of addition, subtraction, multiplication, and division and their commutative, associative, and distributive laws. The exclusion of division by zero is properly emphasized here; throughout both books many notes call attention to this exclusion of division by zero.

A good treatment of equivalent equations is found in Chapter III. A complete treatment of factoring, including the factor theorem, appears in Chapter V. The graph is continually used in simultaneous equations. The theory of exponents and radicals is completed in this course. The discussion of formulas as a preparation for later work in physics is emphasized. Radicals are well treated and followed by a good collection of problems involving radicals.

The topics covered in the two books fully meet the present college requirements with additional chapters on variation, logarithms, arithmetic and geometric progressions, and the binomial theorem for positive integral exponents.

The following definition of irrational number seems unnecessarily complicated and incomplete: "If a number is not the kth power of an integer or a fraction, but if its kth root can be approximated by means of integers and fractions to any specified degree of accuracy, then such a kth root is called an irrational number" (page 70). The incompleteness is willful, for the authors later remark, "There are other kinds of irrational numbers besides those here defined." No *explicit* definition of rational numbers was found. There seems to be a slight inconsistency between this discussion of numbers on page 70 and that of roots on page 11.

We regret to see no use of the symbol i for the complex unit, and the omission of the term *discriminant* in the discussion of the quadratic. An index of definitions would be a useful addition to both books. There is no appendix. The authors have had the courage to omit entirely topics which they believe have no place in a high school algebra, such as simultaneous equations of more than three unknowns (a single example in four

unknowns is given) inequalities, complicated complex fractions.

Judged as a whole, these books are the best texts on high school algebra we have seen, and their use will produce stronger students than we are now receiving from the high schools.

ERNEST B. LYTLE.

Einführung in die Hauptgesetze der zeichnerischen Darstellungsmethoden. By Professor ARTUR SCHOENFLIES. B. G. Teubner, Leipzig and Berlin, 1908. iv + 92 pages. 98 figures.

DURING the last few years the need for more systematic and comprehensive instruction in geometric drawing has become much more keenly felt among school and college teachers. While the courses in descriptive geometry for technical students are frequently sufficiently extensive to claim nearly half the time of the student for a year or more, the corresponding training for prospective teachers is usually very inadequate. Among the attempts to supply this need, numerous recent publications of the firm of B. G. Teubner are worthy of careful consideration. The question may be fairly put, whether the emphasis should be placed on the technical details or on the geometric principles underlying them. The little book of Schütte* is of the first kind; that of Müller and Presler† and also that of Schüssler‡ are much more comprehensive and combine considerable instruction in geometry with the explanations of the constructions. But the book under review puts most of the emphasis on the geometric principles. It is meant for much more mature students than the other books mentioned. The aim is frequently not so much the acquisition of sufficient knowledge or skill to introduce all the geometric details into a figure, but rather the power of making a good illustration by means of a few lines drawn free hand.

After a good discussion of the principles of plane perspective the theorem that any two projective pencils have one right angle formed by corresponding lines is proved in great detail. This theorem is used later. The treatment of the methods of descriptive geometry is brief, but clear. A reader of fair ability and much patience could read these parts profitably

* See review in BULLETIN, vol. 14, p. 294.
† See BULLETIN, vol. 10, p. 207.
‡ Reviewed in BULLETIN, vol. 12, p. 361.

without a teacher, though too few exercises are given to prove his interpretations or measure his skill. Now follow in rapid review the outlines of four important methods, though much too brief to be of maximum service. The treatment of axonometry is confined almost entirely to the proof, elegant in itself, of Polke's theorem that any plane quadrilateral is similar to some plane (parallel) projection of a given tetrahedron. Nine interesting applications are hastily sketched. The determination of apparent contour is the most interesting part of the book; the applications to the anchor ring and hyperboloid of revolution could hardly be followed by a reader to whom these figures are new. Stereographic projection is theoretically discussed, the invariant property of circles and angles being established. The depiction of the edges of a cube on a plane perpendicular to a diagonal is the only illustration. Finally, a few pages are devoted to relief perspective, generalizing for three dimensions what was done earlier in the book for two. A number of instructive notes are added, more fully explaining various points in the text.

The only typographical errors that would cause confusion are on page 23, where regions II, III should be interchanged, and on pages 32, 33, where the letters F_0, F_2 in the text refer to P_0, P_2 respectively in the figures.

VIRGIL SNYDER.

La Géométrie Analytique Générale. Par H. LAURENT. Paris, A. Hermann, 1906. vii + 151 pp.

MANY text-books, chiefly on analysis and algebra, have been written by M. Laurent during the last twenty-five years. The present little volume, with its far-reaching title, suggests that the author is turning his attention, for the moment, to the field of geometry. One soon finds that the underlying thought is that geometry is merely a branch of the theory of numbers. The author has certainly succeeded admirably in emphasizing the fact that geometry may be considered as a purely abstract subject capable of treatment by analytical methods without reference to space perceptions. However, the work is not, as a whole, just what one would desire as a brief first book on the subject of "General analytical geometry." For instance, one feels that much of the strength of the book has been lost by spending an undue amount of time on the one topic of orthogonal substitutions and by delaying the definition of a group too long.

After a three page introduction on trigonometry (where the ratios are defined analytically), the reader is plunged at once into the main topic of the book — orthogonal substitutions. In the conclusion half a dozen pages are devoted to some general remarks about groups of substitutions. A definition of a group is given, and the statement is made that the orthogonal substitutions form a group. In the next paragraph the author says that all possible geometries are, fundamentally, the study of groups of substitutions and their invariants. Should this most important fact be relegated to the conclusion and there passed over lightly? May not the student regard it as a mere afterthought? Is it one of the conclusions that have necessarily arisen in his mind during the study of this book? Indeed, one fears that he would not have sufficient breadth of view to appreciate the meaning of the sentence.

Another criticism that must be made is in regard to the absence of any trace of a bibliography. It may be unnecessary to give references in secondary school books (though one can certainly raise the question even there); but it does seem most unwise to omit from a text of this nature all reference to the literature of the subject. E. B. COWLEY.

Tafeln unbestimmter Integrale. Von G. PETIT BOIS, Bergingenieur in Lüttich. Leipzig, Teubner, 1906. 4to. xii + 154 pp.

THE book contains 2,500 or more indefinite integrals, unnumbered. This is five times as many as occur in B. O. Peirce's Short Table of Integrals, but the latter would be, in general, the more useful book on account of its containing also definite integrals and many auxiliary formulas. In the three-page index to the Tafeln are given the 110 groups into which the formulas are divided. About three-fourths of the integrals are of algebraic functions. As an introduction, there are 49 transformation formulas. The books most used in the compilation were those of Schubert, Minding, Sohncke, Frenet, Graindorge, Brahy, Gregory, Roberts, and Carr. EDWARD L. DODD.

Annuaire du Bureau des Longitudes pour l'An 1909. Paris, Gauthier-Villars.

THE volume of the Annuaire for the current year is without the chemical and physical constants and certain astronomical tables which are inserted in the even-numbered years. As usual

in each successive issue, some of the information is brought up to date; a significant change noted this year is the suppression of the interest tables for $5\frac{1}{2}$ and 6 per cent., and the insertion of those for $1\frac{1}{2}$ and 2 per cent.

The summaries of progress in special lines which we are accustomed to expect deal this year with variable stars by M. G. Bigourdan and with the movements and deformations of the crust of the earth by M. Ch. Lallemand, the former in 116, and the latter in 57 pages. Both of these articles are illustrated by numerous figures and really amount to brief treatises from which it is easy to obtain a fair idea of the theorems, methods and results in the two subjects. Two brief obituary notices of M. Janssen by MM. Radau and Deslandres, and a full alphabetical index close the volume. ERNEST W. BROWN.

NOTES.

THE twenty-fourth regular meeting of the Chicago Section of the AMERICAN MATHEMATICAL SOCIETY will be held at the University of Chicago on April 9–10.

THE Deutsche Mathematiker-Vereinigung now includes 725 members, of whom 60 are Americans. The Circolo Matematico di Palermo has a membership of 635, of whom 105 are Americans.

UNIVERSITY OF PARIS. The following mathematical courses, each two hours per week, are given in the second semester, beginning March 1: — By Professor E. PICARD: Certain developments in series occurring in analysis and mathematical physics. — By Professor E. GOURSAT: Differential equations. — By Professor P. PAINLEVÉ: General laws of motion of systems, analytic mechanics, hydrostatics and hydrodynamics. — By Professor P. APPELL: Elements of analysis and mechanics. — By Professor P. ANDOYER: Mathematical astronomy. — By Professor J. BOUSSINESQ: Thermodynamics and refrigeration. — By Professor G. KOENIGS: General theory of mechanisms.

Semiweekly conferences are held by Professors Raffy, Andoyer, and Servant.

The courses in the Ecole Normale are continuations of those of the first semester (see December BULLETIN, page 148).

UNIVERSITY OF MUNICH.— By Professor F. LINDEMANN: Analytic geometry of space, five hours; Definite integrals and Fourier series, four hours; Seminar, two hours.— By Professor A. VOSS: Integral calculus, five hours; Differential geometry of curved surfaces, II, three hours; Seminar, two hours.— By Professor A. PRINGSHEIM: Algebra, II, four hours; Applications of elliptic functions, three hours.— By Professor A. SOMMERFELD: Optics, four hours; Seminar, two hours.— By Professor H. BRUNN: Elements of higher mathematics for general students, three hours.— By Professor K. DOEHLEMANN: Descriptive geometry, II, with exercises, six hours; Geometric mechanics, with exercises, four hours.— By Dr. F. HARTOGS: Analytic functions of several variables, two hours. — By Dr. O. PERRON: Differential and integral calculus, with exercises, six hours.

UNIVERSITY OF STRASSBURG.— By Professor H. WEBER: Definite integrals and introduction to the theory of functions, four hours; Algebraic numbers, two hours; Seminar, two hours.— By Professor F. SCHUR: General theory of curves and surfaces, four hours; Geometry of movement, two hours; Seminar, two hours.— By Professor J. WELLSTEIN: Introduction to the theory of finite discrete groups, four hours; Riemann surfaces, two hours; Seminar, two hours.— By Professor H. E. TIMERDING: Analytic geometry of space, four hours; Introduction to higher analytic geometry of the plane, four hours; Axonometry and perspective, two hours.— By Professor P. EPSTEIN: Harmonic functions and Bessel functions, four hours.

The following graduate courses in mathematics are announced for the year 1909–1910.

PRINCETON UNIVERSITY. (Each course is three hours per week.) —By Professor H. B. FINE: Advanced algebra (first term). — By Professor H. D. THOMPSON: Coordinate geometry. — By Professor J. H. JEANS: Mechanics and kinematics. —By Professor L. P. EISENHART: Differential geometry. — By Professor O. VEBLEN: Projective geometry, I; Projective geometry, II; Theory of functions of real variables. — By Professsor G. D. BIRKHOFF: Differential equations; Advanced calculus. — By Dr. E. SWIFT: Theory of functions of a complex variable.

PROFESSOR G. BAGNERA, of the University of Messina, has accepted the professorship of analysis at the University of Palermo.

PROFESSOR G. FUBINI, of the University of Genoa, has been appointed professor of mathematical analysis at the technical school at Turin.

DR. R. v. MISES has been appointed docent in mechanics at the technical school at Brünn.

DR. KRYGOWSKY, of the technical school at Lemberg, has been promoted to an associate professorship of mathematics.

PROFESSOR J. PLEMELJ, of the University of Czernowitz, has been promoted to a full professorship of mathematics.

PROFESSOR H. POINCARÉ, of the University of Paris, has been elected honorary member of the academy of sciences at Vienna.

DR. E. BUHL, of the University of Toulouse, has been promoted to a professorship of mathematics.

PROFESSOR M. DISTELI, of the technical school at Dresden, has accepted the professorship of mathematics in the technical school of Carlsruhe made vacant by the resignation of Professor F. SCHUR, who was called to the University of Strassburg.

PROFESSOR TH. REYE, professor emeritus of the University of Strassburg, has been decorated with the order of the red eagle of the second class with the oaken wreath.

PROFESSOR MAURICE LÉVY, of the Collège de France, has been made professor emeritus.

PROFESSOR E. LANDAU, of the University of Berlin, has accepted a professorship of mathematics at the University of Göttingen, as successor to the late Professor H. Minkowski.

DR. P. CARRUS, of the University of Besançon, has been promoted to an associate professorship of mathematics.

MR. D. D. LEIB has been appointed instructor in mathematics at Yale University.

PROFESSOR G. MORERA, of the University and technical school at Turin, died February 8, at the age of 53 years. He was a member of the academy of sciences of Turin, and of the royal academy dei Lincei.

NEW PUBLICATIONS.

I. HIGHER MATHEMATICS.

BAYES (T.). Versuch zur Lösung eines Problems der Wahrscheinlichkeits rechnung. Herausgegeben von H. E. Timerding. (Ostwald's Klassiker der exakten Wissenschaften, 169.) Leipzig, Engelmann, 1908. 8vo. 59 pp. M. 1.20

BÔCHER (M.). An introduction to the study of integral equations. Cambridge, University Press, 1909. 8vo. 8 + 72 pp. (Cambridge tracts in mathematics and mathematical physics, No. 10.) 2s. 6d.

CASTELLS VIDAL (P.). Resumen de las lecciones de análisis matemático, hasta las aplicaciones geométricas del cálculo diferencial. 3a edición. Barcelona, 1908. 737 pp. P. 35.00

DÖLP (H.). Die Determinanten nebst Anwendung auf die Lösung algebraischer und analytisch-geometrischer Aufgaben. Elementar behandelt. 7te Auflage. Darmstadt, Roether, 1908. 8vo. 4 + 95 pp. M. 2.00

ENCYCLOPÉDIE des sciences mathématiques pures et appliquées. Rédigée et publiée d'après l'édition allemande sous la direction de J. Molk. Tome I, Vol. 4. Calcul des probabilités; théorie des erreurs; applications diverses. Rédigé dans l'édition allemande sous la direction de F. Meyer. Fascicule 2. Paris et Leipzig, Teubner, 1908. 8vo. Pp. 161–320. Fr. 5.00

HÜBNER (H.). Ueber den Fermat'schen Satz. Erlangen, Junge, 1908. 8vo. 40 pp. M. 1.50

MATAIX ARACELI (C.). Resumen de las lecciones de cálculo integral y mecánica racional, explicadas en la Escuela Central de Ingenieros Industriales. Vols. I and II. Madrid, 1908. Vol. I, 351 pp.; Vol. II, 626 pp. P. 40.00

MEYER (F.). See ENCYCLOPÉDIE.

MOLK (J.). See ENCYCLOPÉDIE.

PENKMAYER (R.). Beweis des Satzes von Fermat: Die Gleichung $a^n + b^n = c^n$ ist in ganzen Zahlen unlösbar, wenn $n > 2$ ist. München, Lindauer, 1908. 8vo. 7 pp. M. 0.50

TIMERDING (H. E.). See BAYES (T.).

WEITZENBÖCK (R.). Komplex-Symbolik. Eine Einführung in die analytische Geometrie mehrdimensionaler Räume. (Sammlung Schubert, LVII.) Leipzig, Göschen, 1908. 8vo. 6 + 191 pp. Cloth. M. 4.80

II. ELEMENTARY MATHEMATICS.

AMALDI (U.). See ENRIQUES (F.).

BRUMBY (A. G.). Elementary practical geometry. London, Normal Press, 1909. 8vo. 214 pp. 1s. 6d.

COURS d'algèbre et de trigonométrie à l'usage des sous-officiers (élèves officiers). Rédigé conformément au programme de 1905. Paris, Charles-Lavauzelle, 1908. 16mo. 193 pp. Fr. 3.00

DREYFUS (L.). See FORT (L.).

DURELL (C. V.). A course of plane geometry for advanced students. Part 1. London, Macmillan, 1909. 8vo. 232 pp. 5 s.

ENRIQUES (F.) e AMALDI (L.). Elementi di geometria. 3a edizione ridotta. Bologna, Zanichelli, 1908. 16mo. 7 + 295 pp. L. 2.50

FORT (L.) et DREYFUS (U.). Eléments de géométrie, conformes aux programmes de 1905. Classe de 3e B. Avec de nombreux exercices. Paris, Paulin, 1909. 18mo. 201 pp. Fr. 2.50

LEDER (E.). Die Praxis des Logarithmen-Rechnens unter Berücksichtigung der einschlägigen tabellarischen, graphischen und mechanischen Hilfsmittel. Ein Leitfaden für Kaufleute, Techniker, Gewerbetreibende, überhaupt für jeden praktischen Zahlenrechner. Berlin, Kühl, 1908. 8vo. 127 pp. M. 6.00

LUBARSKY (L. H.). Elementary algebra: manual of examinations in mathematics; regents' questions and answers. New York, Engineering News Pub. Co., 1909. 8vo. 286 pp. Cloth. $1.50

NOCIONES elementales de geometría aplicadas al dibujo lineal; por los hermanos de las escuelas christianas. 13a edición. Paris, Procuraduria general, 1908. 16mo. 112 pp.

PATERSON (W. E.). School algebra, Part 2. London, Frowde, 1909. 8vo. 292 pp. Cloth. 2s. 6d. With answers, 3 s.

——. School algebra. Complete. London, Frowde, 1909. 8vo. 604 pp. Cloth. 4 s. With answers, 5 s.

RIBONI (G.). Elementi di geometria, ad uso delle scuole secondari e superiori. 5a edizione, con aggiunte e correzioni. Bologna, Zanichelli, 1908. 16mo. 531 pp. L. 3.00

III. APPLIED MATHEMATICS.

ABRAHAM (M.). Theorie der Elektrizität. Vol. II. Elektromagnetische Theorie der Strahlung. 2te Auflage. Leipzig, Teubner, 1908. 8vo. 12 + 404 pp. Cloth. M. 10.00

BARUCCI (O.). Corso di disegno geometrico e ornamentale, per le scuole tecniche e complementari. Parte I. Torino, Paravia, 1909. 8vo. L. 1.60

BLANCHIN (P.). See ZENNECK (J.).

CRAMP (W.) and SMITH (C. F.). Vectors and vector diagrams, applied to the alternating current circuit with examples of their use in the theory of transformers, and of single and polyphase motors, etc. New York, Longmans, 1909. 8vo. 16 + 252 pp. Cloth. $2.50

EINHORN (M.). Maturitäts-Aufgaben aus der darstellenden Geometrie, geeignet als Uebungs- und Repetitions-Aufgaben für die oberen Klassen der Realschule und für die Vorbereitung zur Aufnahme-Prüfung am Eidgenössischen Polytechnikum. 1tes Heft. Zürich, Speidel, 1908. 8vo. 30 pp. M. 0.70

FÖPPL (A.). Vorlesungen über technische Mechanik. (In 6 volumes.) Vol. 4: Dynamik. 3te, stark veränderte Auflage. Leipzig, Teubner, 1909. 8vo. 8 + 422 pp. Cloth. M. 10.00

GAUSS. See LAGRANGE.

GIBSON (A. H.). Hydraulics and its application. New York, Van Nostrand, 1908. 8vo. 16 + 757 pp. Cloth. $5.00

GUÉRARD (G.). See ZENNECK (J.).

HANCOCK (E. L.). Applied mechanics for engineers: a textbook for engineering students. New York, Macmillan, 1909. 12mo. 11 + 385 pp. Cloth. $2.00

JACOBI. See LAGRANGE.

JOURDAIN (P. E. B.). See LAGRANGE.

LAGRANGE, RODRIGUES, JACOBI und GAUSS. Abhandlungen über die Prinzipien der Mechanik, herausgegeben von P. E. B. Jourdain. (Ostwald's Klassiker der exakten Wissenschaften, 167.) Leipzig, Engelmann, 1908. 8vo. 68 pp. M. 1.40

PICOT (E.). See ZENNECK (J.).

RODRIGUES. See LAGRANGE.

SMITH (C. F.). See CRAMP (W.).

STEINMETZ (C. P.). Theory and calculation of alternating current phenomena. 4th edition. London, Spon, 1909. 8vo. Cloth. 21 s.

THUET (J.). Cours de mécanique à l'usage des élèves des classes de mathématiques A et B et des candidats à l'École navale et à l'École de Saint-Cyr (programme de 1905). Paris, Relin, 1908. 8vo. 360 pp. Fr. 4.50

ZENNECK (J.). Les oscillations électromagnétiques et la télégraphie sans fil. Ouvrage traduit de l'allemand par P. Blanchin, G. Guérard, et E. Picot. Vol. 2: les oscillations ouvertes et les systèmes couplés; les ondes électromagnétiques et la télégraphie sans fil. Paris, Gauthier-Villars, 1908. 8vo. 496 pp. Fr. 17.00